6. Chern Insulators: The Qi-Wu-Zhang Model

Most important chapter: heart of topological insulators.

• Required: Thouless pumping
• New theory tool: Promoting time $t \rightarrow$ quasimomentum $k$
• Main results: Edge states in two-dimensional systems
  Bulk Chern number predicts edge states
  Topological protection

• Toy model: Qi-Wu-Zhang (obtained from Thouless pump
  in Rice-Mele by promoting $t \rightarrow k$
Reminder 1: Thouless pump sequence, Rice-Mele

Pump charge along a dimerized chain using sublattice potential:

\[ \hat{H}(k,t) = d(k,t)\hat{\sigma} = (v(t) + w(t)\cos k)\hat{\sigma}_x + w(t)\sin k\hat{\sigma}_y + u(t)\hat{\sigma}_z \]

\[ u(t) = \sin \Omega t; \]
\[ v(t) = \bar{v} + \cos \Omega t; \]
\[ w(t) = 1, \]

Graph showing the amplitudes of \( u, v, w \) over time, with \( d_z, d_y, d_x \) arrows indicating the direction of the sublattice potential.
Reminder 2: Topologically Protected Edge States in Thouless pump

Topologically protected = robust:
1) Time - Periodic drive
2) No long range hopping

1→ spectrum time-periodic
2→ spectrum continuous
2→ bulk gap separates two edges
→ no direct coupling, 
→ crossing, not anticrossing
Reminder 3: Thouless pump in the bulk in d-space:
# times origin in torus = # charge pumped = Chern #

control freak sequence:

smooth sequence:
Reminder 4: Net number of charge pumped up in energy at an edge is protected against continuous deformations.
New material, class 6:  
From Thouless pump to Chern insulator
Promote time $t \rightarrow$ wavenumber $k$

1D time-periodic Rice-Mele $\rightarrow$ 2D Qi-Wu-Zhang

$$\hat{H}_{RM}(k, t) = \sin k \hat{\sigma}_y + \sin \Omega t \hat{\sigma}_z + (\bar{v} + \cos \Omega t + \cos k) \hat{\sigma}_x$$

$$\Omega t \rightarrow k_y$$

$k \rightarrow k_x$

$\hat{\sigma}_y \rightarrow \hat{\sigma}_x$

$\hat{\sigma}_z \rightarrow \hat{\sigma}_y$

$\hat{\sigma}_x \rightarrow \hat{\sigma}_z$

$\bar{v} \rightarrow u$

$$\hat{H}_{QWZ}(k_x, k_y) = \sin k_x \hat{\sigma}_x + \sin k_y \hat{\sigma}_y + (u + \cos k_x + \cos k_y) \hat{\sigma}_z$$

2D square lattice, nearest-neighbor spin-dependent hopping
Edge states rising/falling in Thouless pump → unidirectional edge modes in Chern insulators

Topologically protected = robust:
- No long range hopping

→ spectrum periodic & smooth
→ bulk gap separates two edges → no direct coupling → crossing, not anticrossing
Presence, net # of edge state modes seen in bulk: # times origin in torus = # edge state modes = Chern #
Net number of clockwise-propagating edge state modes in the gap is protected against continuous deformations.
Net edge states at some section of edge $\rightarrow$ edge states all around (unitarity $\rightarrow$ particles cannot accumulate)

Topologically protected = robust against:

- Arbitrary disorder on edges
- Some disorder in bulk
  (interesting variation on Anderson localization)
Summary: Chern Insulators have robust edge states predicted by bulk Chern #

- **Required:** Thouless pumping (ensure edge states, Chern #)
- **Theory tool:** Promote \( t \rightarrow \text{quasimomentum } k \)
- **Main results:**
  - Edge states in two-dimensional systems
  - Bulk Chern number predicts edge states
  - Topological protection due to no backscattering
  - Robust against disorder (large edge, small bulk)
- **Toy model:** Qi-Wu-Zhang (from Thouless pump Rice-Mele)
  - Tune Chern number by onsite magnetic field \( u \) (-2, 0, 2)

\[
\hat{H}_{QWZ}(k_x, k_y) = \sin k_x \hat{\sigma}_x + \sin k_y \hat{\sigma}_y + (u + \cos k_x + \cos k_y)\hat{\sigma}_z
\]