4π-periodic Josephson effect

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The Josephson junction is a simple electric device where two superconductors are connected by a thin insulating barrier. It exhibits the ac Josephson effect: when biased by a dc voltage, an ac current develops at the Josephson frequency $f_J = 2eV/h$. This is true for conventional, s-wave superconductors. In this lecture, we discuss that this effect changes qualitatively when the Josephson junction is made of p-wave superconductors: the frequency of the ac current halves, it is $f_J/2$ instead of the conventional f_J. This is often called 4π-periodic Josephson effect. We use the Kitaev chain as a model for the p-wave junction, and use it to calculate the current-phase relation. We also discuss some experimental constraints required to observe the 4π-periodic Josephson effect, and introduce an experimental setup where the effect is detected with a superconducting radiation detector.

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I. A VOLTAGE-BIASED S-WAVE JOSEPHSON JUNCTION RADIATES AT THE JOSEPHSON FREQUENCY

1. A simple circuit built around a Josephson junction (JJ) is shown in Fig. 1. The superconducting contacts are s-wave superconductors. A dc bias voltage V is switched on. Then, an ac current flows through the junction, characterized by the Josephson frequency $f_J = 2eV/h$, corresponding to $f_J/V \approx 483.6\,\mathrm{MHz}/\mu\mathrm{V}$. This is called the *ac Josephson effect*. Such an ac current induces electromagnetic radiation with frequency f_J, which depends on the voltage V applied across the junction. With a frequency-sensitive detector, the radiation can be observed, and its frequency can be determined.

2. Aside: an s-wave junction can radiate also at the higher harmonics of the Josephson frequency, nf_J ($n = 1, 2, \dots$). We can disregard that feature for this lecture.

3. Our goal in this lecture is to illustrate that the ac Josephson effect can in principle be used to experimentally distinguish between s-wave and p-wave superconductivity, that is, between topologically trivial and nontrivial phases of a one-dimensional superconductor. In particular, we illustrate that a p-wave junction can be distinguished from an s-wave junction, because the p-wave junction radiates at half of the Josephson frequency $f_J/2$. This effect is also called the 4π*-periodic ac Josephson effect*, for reasons to be clarified below.

4. We illustrate the s-wave case using a minimal model, where each superconducting electrode of the junction is represented by a single-site superconductor. The two electrodes are assumed to be identical, with the only

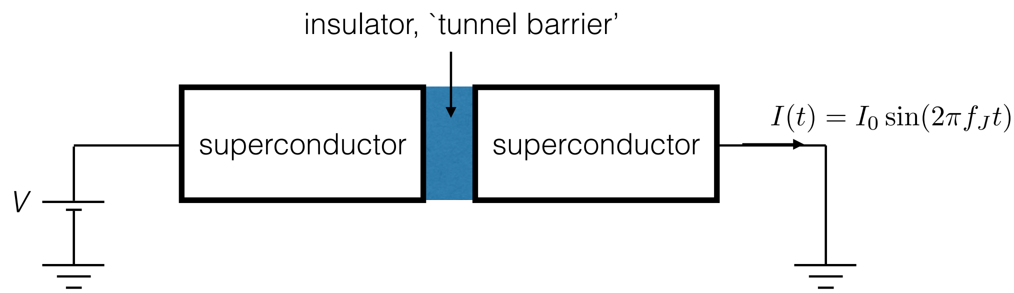

FIG. 1. **An s-wave Josephson junction exhibiting the ac Josephson effect.**

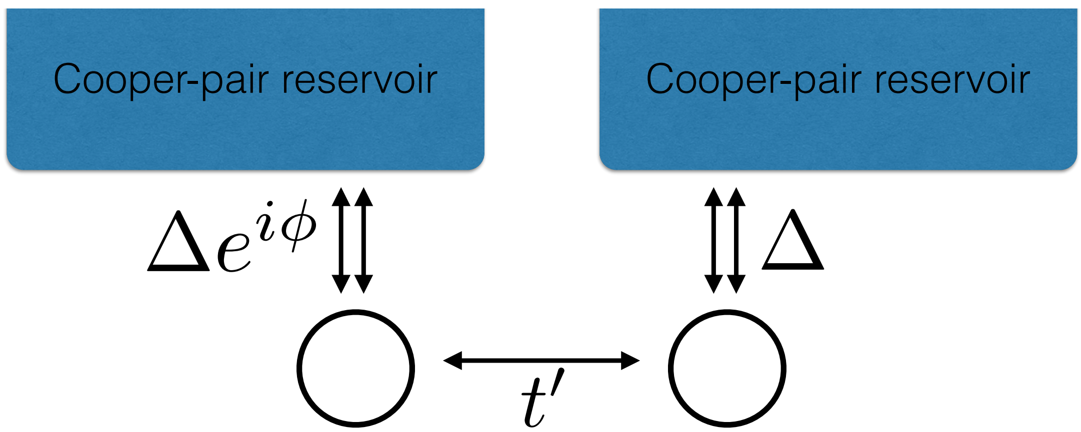

FIG. 2. **Structure of the two-site model Hamiltonian of the s-wave Josephson junction.**

difference being their phases of their superconducting order parameters. The model Hamiltonian reads

$$H = H_L + H_R + H_t, \tag{1}$$
$$H_L = \Delta e^{i\phi} c^\dagger_{L\uparrow} c^\dagger_{L\downarrow} + h.c., \tag{2}$$
$$H_R = \Delta c^\dagger_{R\uparrow} c^\dagger_{R\downarrow} + h.c., \tag{3}$$
$$H_t = t' \sum_{s\in\{\uparrow,\downarrow\}} c^\dagger_{Ls} c_{Rs} + h.c. \tag{4}$$

Note that $\Delta > 0$ and we assume $t' > 0$ for concreteness. The *phase difference* or *phase bias* is denoted by ϕ. We accept the voltage-phase Josephson relation as a starting point, $\dot{\phi} = 2eV/\hbar$. The structure of this Hamiltonian is illustrated in Fig. 2.

5. *Exercise.* Write out the Fock-space Hamiltonian matrix of H_L, and calculate its energy eigenstates and eigenvalues.

6. Interestingly, we will see that at zero-temperature equilibrium, there is a finite current flowing through the tunnel barrier of the junction, if the phase bias is not zero or π. More precisely, we will show the current-phase Josephson relation (for a specific case) $I(\phi) = I_0 \sin(\phi)$. To obtain the time dependence of the current, we will use an adiabatic picture: we will assume that the voltage V is small and dc, hence the phase bias has a slow time dependence $\phi(t) = 2eVt/\hbar$, and in any time t, the junction is in its instantaneous zero-temperature equilibrium state (ground state) determined by $\phi(t)$. As a result, we have $I(t) = I_0 \sin(2eVt/\hbar)$, that is, $I(t) = I_0 \sin(2\pi f_J t)$, so, indeed, the voltage-biased junction produces electromagnetic radiation at the Josephson frequency.

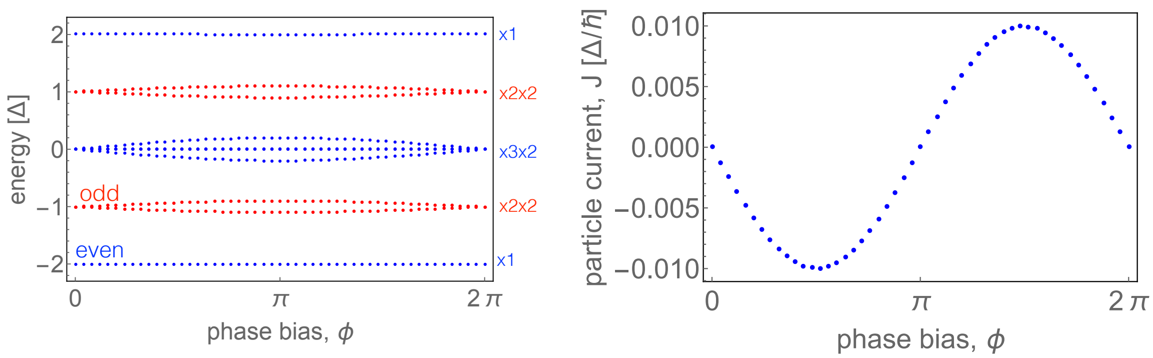

FIG. 3. **Energy spectrum (left panel) and ground-state current (right panel) of an s-wave Josephson junction.**

7. To demonstrate the preceding statement, consider now the ground state of our two-site junction Hamiltonian H for a finite, static phase bias ϕ. We claim that the current flowing through the junction is finite. To show that, we need the particle current operator J representing electron flow through the junction, and need to calculate $\langle\Psi_0|J|\Psi_0\rangle$. One way to obtain the operator J is to consider the time-dependence of the particle number on site R: $Q_R = \sum_s c^\dagger_{Rs} c_{Rs}$, and consider the net particle flux into this site

$$J_R \equiv \dot{Q}_R = -\frac{i}{\hbar}[Q_R, H] = -\frac{i}{\hbar}[Q_R, H_R] - \frac{i}{\hbar}[Q_R, H_t] \equiv J_{R\Delta} + J_{Rt}. \tag{5}$$

Clearly, the ground-state average of this quantity is zero, $\langle\Psi_0|J_R|\Psi_0\rangle = 0$, since the ground state is a stationary state, so no physical quantity can change in time when this state is realized. However, the current flowing through the junction is not J_R, but only J_{Rt}: the other contribution $J_{\Delta t}$ represents Cooper pairs being exchanged between site R and the Cooper-pair reservoir attached to that site. This is also seen in Fig. 2. Hence, $J = J_{Rt}$, and we need to evaluate $\langle\Psi_0|J_{Rt}|\Psi_0\rangle$. Note also that the exchange of electrons between the two sites is always compensated by the exchange of electrons between site R and its Cooper-pair reservoir; that is, since the ground state $|\Psi_0\rangle$ is stationary ($\langle\Psi_0|J_R|\Psi_0\rangle = 0$), therefore $\langle\Psi_0|J_{Rt}|\Psi_0\rangle = -\langle\Psi_0|J_{R\Delta}|\Psi_0\rangle$.

8. Fig. 3 shows the energy spectrum and the ground-state current of our two-site junction, as functions of the phase bias ϕ, for 'weak tunneling', $t'/\Delta = 0.1$. Note that we use colors to distinguish energy eigenstates with even (blue) and odd (red) fermion parity. The ground state is nondegenerate and has even parity. The current reproduces the sinusoidal dependence promised above. The amplitude of the current is $J_0 = \frac{t'^2}{\hbar\Delta}$; for the parameter values $t' = 50\,\mu$eV, $\Delta = 500\,\mu$eV, this translates to an electric current amplitude of $I_0 = eJ_0 \approx 1.22$ nA.

9. *Exercise.* The results shown in Fig. 3 are obtained numerically. They can also be confirmed analytically. For example, one can do perturbation theory in the small parameter t'/Δ to obtain the ground state of the junction, and to calculate the current from that. A further alternative is to calculate the ground-state current in terms of the Bogoliubov-de Gennes representation. These are instructive exercises for the interested reader.

10. From the above exercise, some of the 'physics' behind the ac Josephson effect is also clarified. For example, it turns out that the Josephson current is carried by *Cooper-pair tunneling*: the elementary charge-transfer process is actually a two-step process. First, a Cooper-pair on one side is broken up, one of the two electrons is transmitted to other side, while the other remains as a quasiparticle. This is a virtual intermediate state that has an energy penalty of $\approx \Delta$. Second, the quasiparticle is also transmitted and they form a Cooper pair together on the other side, hence their final energy is the same as the energy of the initial state. The fact that the Josephson current is due to Cooper-pair tunneling is also reflected in above relation $J_0 \sim t'^2$.

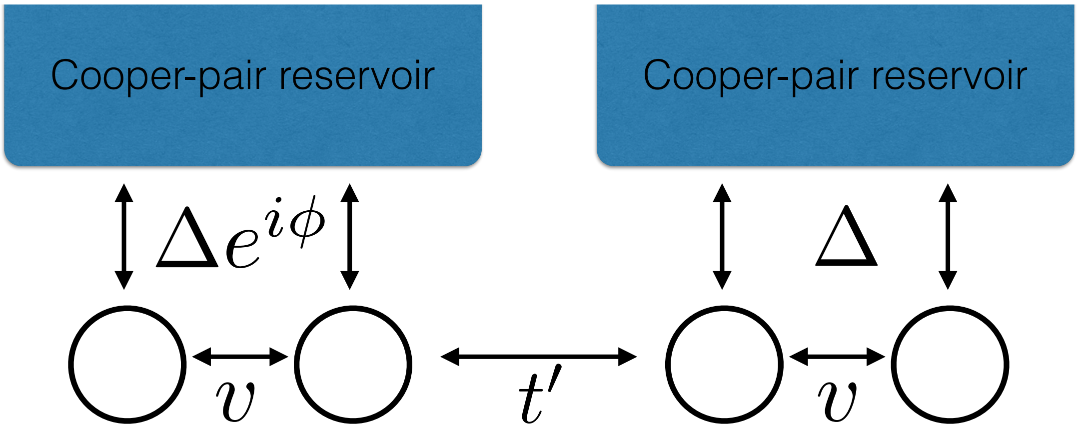

FIG. 4. **Structure of the four-site model Hamiltonian of the p-wave Josephson junction.**

II. A VOLTAGE-BIASED P-WAVE JOSEPHSON JUNCTION RADIATES AT HALF OF THE JOSEPHSON FREQUENCY

1. Calculating the current for a p-wave junction is conceptually similar to that for the s-wave case, but there are also some qualitative differences on the conceptual level. Most importantly, the ground state of a p-wave junction is not separated from the excited state by the superconducting gap Δ; instead, there are 4 low-energy levels, energetically close to each other. Therefore, it is not straightforward to adopt the 'adiabatic picture' followed in the case of the s-wave junction.

2. The minimal model for the p-wave junction consists of two Kitaev double dots in the topological fully dimerized limit. The Hamiltonian is illustrated in Fig. 4, and in the topological fully dimerized limit $v = \Delta$ it reads

$$H_L = \Delta(c_1^\dagger c_2 + h.c.) + \Delta(e^{i\phi} c_1^\dagger c_2^\dagger + h.c.), \tag{6}$$

$$H_R = \Delta(c_3^\dagger c_4 + h.c.) + \Delta(c_3^\dagger c_4^\dagger + h.c.), \tag{7}$$

$$H_t = t'(c_2^\dagger c_3 + h.c.). \tag{8}$$

3. Fig. 5 left panel shows the full energy spectrum of the p-wave junction as the function of the phase bias ϕ, for a weak tunneling amplitude $t'/\Delta = 0.1$. Fig. 5 right panel shows a zoom-in on the four lowermost energies. Again, we use colors to indicate even (blue) and odd (red) fermion parity. Key features of the low-energy spectrum are as follows. (i) There are four levels in the low-energy part, forming two twofold degenerate pairs. (ii) For both pairs, one state of the pair is even (blue), the other is odd (red). (iii) At phase bias $\phi = \pi$, all four levels are degenerate.

4. Discussion of (i): For a disconnected junction, $t' = 0$, we have two topological Kitaev double dots, each of them having a twofold degenerate ground state. That is, a disconnected junction would have a fourfold degenerate ground state for any ϕ. Here, the weak tunneling induces a small splitting of this fourfold degeneracy for a generic ϕ, but these four levels remain close to each other, as seen in the right panel of Fig. 5.

5. Discussion of (ii): The even-odd degeneracy of both pairs is explained by the fact that even in the tunnel-coupled junction, there are two localized Majorana zero modes, call them γ_{LL} and γ_{RR}, at the very ends of the junction, on site 1 and on site 4. These Majorana zero modes remain Majorana zero modes even if sites 2 and 3 are coupled by the tunneling amplitude t' and form a finite-energy fermionic excitation. Therefore, γ_{LL} and γ_{RR} define a single zero-energy fermionic excitation, $d_{\text{end}} = \frac{1}{\sqrt{2}}(\gamma_{LL} + i\gamma_{RR})$, which guarantees that all energy eigenvalues have the twofold even-odd degeneracy seen in Fig. 5.

6. Discussion of (iii): The energy splitting between the pairs is interpreted as the tunneling-induced hybridization of the two central Majorana zero modes γ_{LR} and γ_{RL} of the disconnected junction. Due to the nonzero tunneling t', these two Majorana zero modes are no longer eigenvectors of the Bogoliubov-de Gennes matrix, but their appropriate linear combination does give a finite-energy fermionic excitation. A remarkable exception is the case $\phi = \pi$, see Fig. 5, where this fermionic excitation energy takes a zero value.

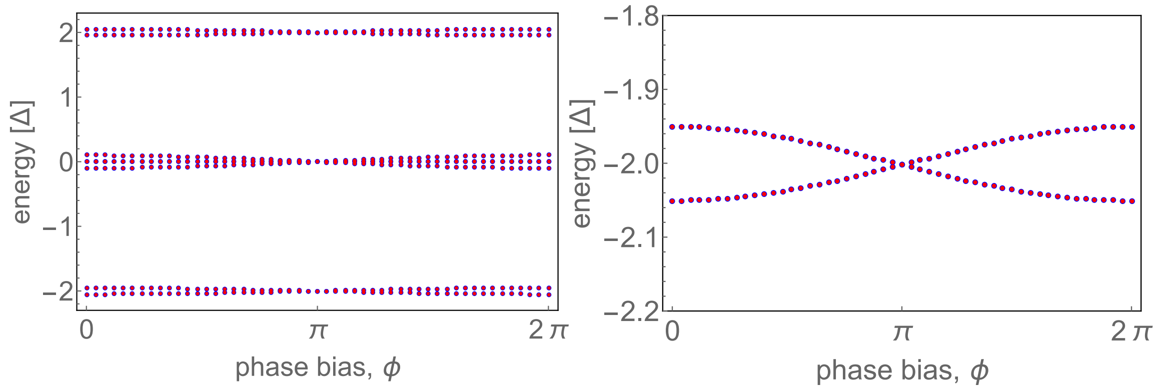

FIG. 5. **Full energy spectrum (left panel) and the low-energy part of the energy spectrum (right panel) of the four-site p-wave Josephson junction.**

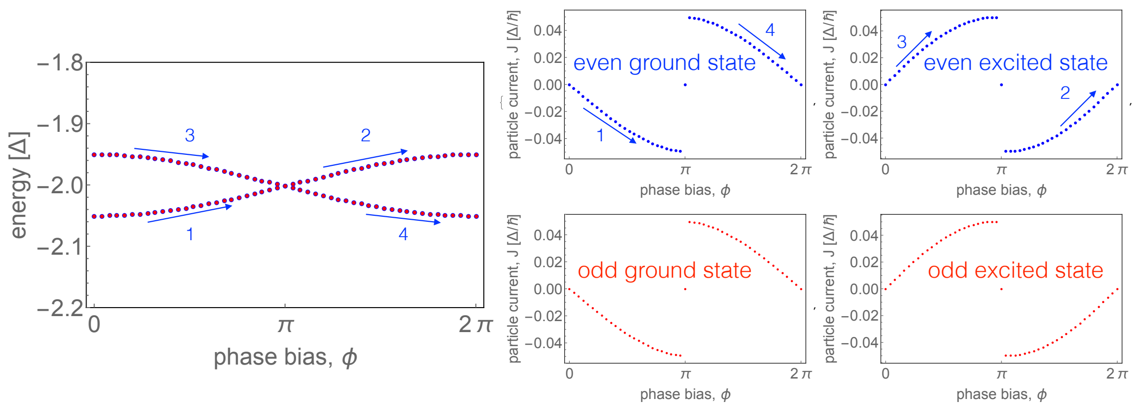

FIG. 6. **Low-energy spectrum showing four energy eigenstates (left panel), and the phase-dependent current for each energy eigenstate.**

7. *Exercise.* Prove that for $\phi = \pi$, the fermionic excitation composed of γ_{LR} and γ_{RL} has zero energy, and generalize this result to longer chains, using the Bogoliubov-de Gennes representation and perturbation theory in the tunnel t'.

8. The current flowing through the junction can be calculated the same way as for the s-wave case, for each of these four low-energy states. The four resulting current-phase relations are shown in Fig. 6. The current amplitude is $J_0 = t'/(2\hbar)$, and the phase dependence of the current is sinusoidal – these can be derived analytically using first-order degenerate perturbation theory in t'/Δ. Note that the current is the same in the even and odd ground states. That is no surprise: we have seen (lecture 5, section II.) that local observables in two states are identical if those two states differ only by a zero-energy fermionic excitation built from Majorana zero modes (that is d_{end} in this case).

9. The question is: assume that initially, the phase bias is $\phi = 0$, the junction is in its even ground state, and a voltage V across the junction is switched on; how does the current depend on time?

10. Assuming no quasiparticle poisoning, and using the adiabatic picture as before, we conclude that the evolution of the state will certainly follow the instantaneous even ground state up to $\phi = \pi$, that is, up to $t = h/(eV)$. From that time point, how does the time evolution go on? Will the state evolve as the even ground state, or as the even excited state? It is a generic feature of such level crossings that the state evolution follows the excited state after the crossing is left behind (cf. the Landau-Zener problem). But we can also come to this conclusion just by looking at the current-phase relations: it is natural to expect that any physical quantity, including the particle current, should be a continuous function of time, and that implies that from $\phi = \pi$ the state evolution will follow the even excited state.

11. The preceding argument implies that the evolution of the state will follow the numbering shown in the left panel of Fig. 6. Namely, first the state will follow the even ground state (1, blue dots) for $\phi \in [0, \pi]$, that is, for $t \in [0, h/(4eV)]$. Then, it will follow the even excited state, see (2, blue dots) and (3, blue dots), for $t \in [h/(4eV), 3h/(4eV)]$. Finally, it will follow the even ground state (4, blue dots) for $t \in [3h/(4eV), h/(eV)]$. This means that the state evolution will be periodic in time, but not with the Josephson time period $h/(2eV)$, but with twice the Josephson time period, that is, $h/(eV)$. Of course, the time dependence of the current will also have twice the Josephson time period, that is, $h/(eV)$.

12. Therefore we conclude that the time evolution of the current is $I(t) = -I_0 \sin(eVt/\hbar) = -I_0 \sin(2\pi(f_J/2)t)$. Hence, the voltage-biased p-wave junction radiates with half of the Josephson frequency.

III. DISCUSSION

1. *In a p-wave junction, fast quasiparticle poisoning makes the current-phase relation* 2π*-periodic.* This is the case when quasiparticle poisoning time is shorter than the Josephson time (that is, the inverse Josephson frequency), $t_{\rm qp} \ll h/(2eV)$; for a realistic value of $V = 10\,\mu$V, this translates to $t_{\rm qp} \ll 200$ ps.

2. The adiabatic picture used in the above discussion of the 4π-periodic Josephson effect can break down for various reasons. For example, if the voltage applied between the two superconducting contacts it too large, approaching the superconducting gap as $eV \sim \Delta$, then the resulting time-evolving phase winds so quickly that it can excite the junction from its ground state to excited states.

3. On the other hand, if the voltage is too low, approaching the tunneling energy scale as $eV \lesssim t'$, then the corresponding phase dynamics can induce transitions between the two low-energy states of a given parity, e.g., the two blue-dotted levels in Fig. 6, left panel.

4. In a real setup, there is a small gap opened at the apparent level crossing in Fig. 6, left panel, at $\phi = \pi$. This is due to the fact that the Majorana zero modes have nonzero spatial extents and thereby hybridize with each other. This small gap opens up the possibility of remaining in the even ground state when the phase bias passes through the $\phi = \pi$ point. Quantitatively, the probability of evolving to the excited state (P_{LZ}) and the probability of remaining in the ground state ($1 - P_{LZ}$) is described by the so-called Landau-Zener model, which we do not detail here. We note, however, that $P_{LZ} = 1$ in case of a level crossing, and for a finite anticrossing gap, P_{LZ} decreases (i) if the gap size is increased, and (ii) if the phase-winding speed is decreased. Clearly, if P_{LZ} is significantly less than 1, then the 4π-periodicity of the Josephson effect is lost, and a 2π-periodicity is reinforced for $P_{LZ} \ll 1$.

5. The work of Laroche et al. claims to demonstrate the 4π Josephson effect, using an experimental setup where the Josephson radiation is detected by an appropriately designed frequency-sensitive radiation detector. Below, we briefly summarized the principle of the experiment.

6. Interestingly, the detector itself is an s-wave Josephson junction. The measured quantity at the detector is its dc current, I_{det}^{dc}, and not the time-varying Josephson current. For a voltage-biased s-wave Josephson junction, the voltage dependence of the dc current shows an 'switching' or 'activation' behavior: it is essentially zero as long as $eV_{det} < 2\Delta_{det}$, where V_{det} is the voltage applied between the s-wave contacts of the detector, and Δ_{det} is the superconducting gap of the s-wave contacts of the detector, and the dc current switches to a finite value at $eV_{det} = 2\Delta_{det}$.

7. The reason for this switching behavior is explained by the following simple energy conservation argument. In the presence of the bias voltage V_{det}, the excess energy on the left contact is $N_e e V_{det}$, where N_e is the number of electrons there. A single Cooper pair on the left contact can split up, in such a way that one of the electrons is transmitted to the right contact, and the other electron stays on the left as a quasiparticle, if this final state is energetically favorable over the initial state: $N_e e V_{det} \geq [(N_e - 1)eV_{det} + \Delta_{det}] + \Delta_{det}$. On the right hand side, the term [.] is the energy of the left contact in the final state, whereas the rest is the energy of the right contact in the final state, supporting the single quasiparticle. Rearranging this equation yields the above threshold condition, i.e., that the I_{det}^{dc} switches on for $eV_{det} > 2\Delta_{det}$.

8. In the presence of an incoming radiation with frequency f, the electrons of the Cooper pair can gain energy from that radiation, changing the threshold condition to $eV_{det} > 2\Delta_{det} - hf$. Therefore, by measuring the threshold voltage without and with radiation, the difference will reveal the frequency of the radiation. This principle is used to determine the radiation frequency of the Josephson junction that is being probed, and to distinguish between the s-wave junction, which radiates at f_J, and the p-wave junction, which radiates at $f_J/2$.

RESOURCES

Overview:
Topology in Condensed Matter, https://topocondmat.org/w2_majorana/signatures.html
Dynamical effects, e.g., Landau-Zener transitions:
San Jose et al., *ac Josephson Effect in Finite-Length Nanowire Junctions with Majorana Modes*, Phys. Rev. Lett. 108, 257001 (2012)
Pikulin and Nazarov, *Phenomenology and dynamics of a Majorana Josephson junction*, Phys. Rev. B 86, 140504(R) (2012)
Houzet et al., *Dynamics of Majorana States in a Topological Josephson Junction* Phys. Rev. Lett. 111, 046401 (2013).
Experiment:
Laroche et al., *Observation of the* 4π*-periodic Josephson effect in InAs nanowires*, https://arxiv.org/abs/1712.08459