

Lecture 4: Simple quantum information protocols with the Kitaev double dot

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In lectures 2 and 3, we have introduced the Bogoliubov-de Gennes method, and we used that to determine the quasiparticle excitation spectrum of a one-dimensional p-type superconductor, modelled by an infinite Kitaev chain. Here, we consider a finite, two-site Kitaev chain, which we call ‘Kitaev double dot’ (KDD), and discuss how to define a controllable quantum bit in such a system, as well as how to carry out certain simple quantum information protocols – initialization, single-qubit control, readout, dephasing-time measurement – with that qubit. The concept of ‘Majorana zero modes’ and the ‘topological protection’ of qubits based on 1D topological superconductors will be discussed in the next lecture.

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I. THE TOPOLOGICAL FULLY DIMERIZED KITAEV DOUBLE DOT HAS A TWOFOLD DEGENERATE GROUND STATE

1. The Kitaev double dot (KDD), that is, a two-site Kitaev chain, is shown in Fig. 1a. In this figure, the solid line depicts the hopping v , whereas the dotted line depicts the (p-wave) pair potential Δ ; recall that the latter also connects nearest-neighbour sites. The Hamiltonian of the system is:

$$H = \epsilon_1 c_1^\dagger c_1 + \epsilon_2 c_2^\dagger c_2 + v(c_1^\dagger c_2 + c_2^\dagger c_1) + \Delta(c_1^\dagger c_2^\dagger + c_2 c_1). \quad (1)$$

ϵ_1 and ϵ_2 are the on-site energies. Assume $v > 0$ and $\Delta > 0$ for simplicity. The Hamiltonian matrix in the 00, 01, 10, 11 ‘occupation number basis’ reads

$$H = \begin{pmatrix} 0 & 0 & 0 & \Delta \\ 0 & \epsilon_2 & v & 0 \\ 0 & v & \epsilon_1 & 0 \\ \Delta & 0 & 0 & \epsilon_1 + \epsilon_2 \end{pmatrix} \quad (2)$$

2. Note that the even and odd sectors in H are decoupled (which is a generic feature of free fermionic Hamiltonians), and thereby the eigenvalues and eigenvectors can be found analytically by diagonalizing two 2x2 matrices.

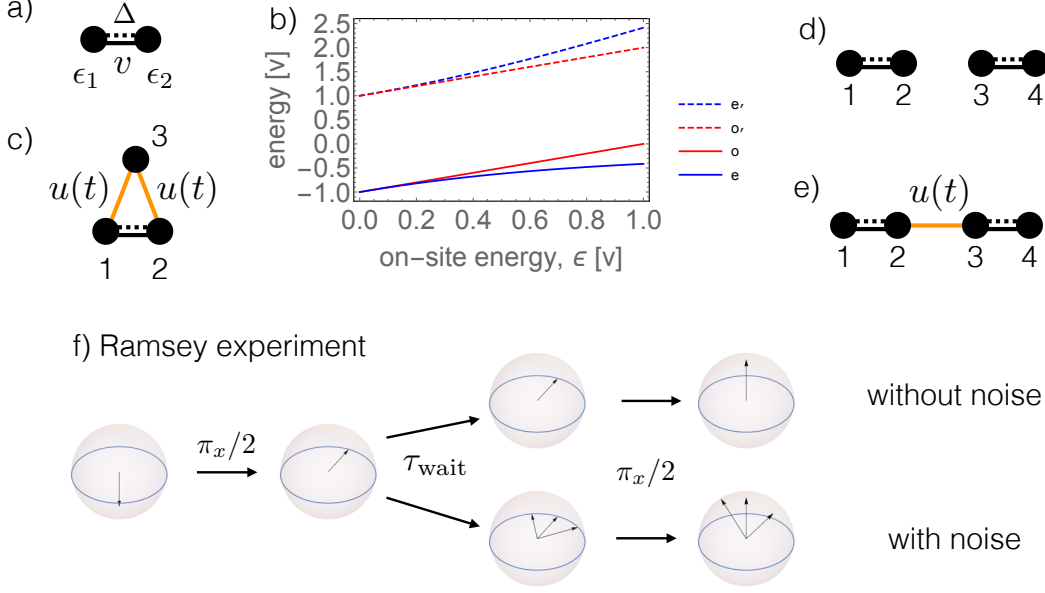


FIG. 1. **Simple quantum information protocols with the Kitaev double dot.** (a) The Kitaev double dot, with on-site energies ϵ_1 and ϵ_2 , hopping v , and pair potential Δ . (b) Energy spectrum of the Kitaev double dot, for $\Delta = v$ and $\epsilon_1 = \epsilon_2 = \epsilon$. (c) Setup for parity-to-charge conversion and parity readout of the Kitaev double dot (sites 1 and 2) using an auxiliary site (3). (d) Two Kitaev double dots to form a Majorana qubit. (e) Setup for non-protected x-rotation of the Majorana qubit. (f) Schematics of a Ramsey experiment of a qubit. It aims to determine the noise-induced inhomogeneous dephasing time T_2^* via measuring the final-state polarization $\langle \sigma_z \rangle(\tau_{\text{wait}})$.

3. As before, we assume that we have experimentally realized this Hamiltonian, and have control over all parameters. We will also discuss what happens when these parameters have some uncontrolled components (disorder/noise).
4. The parameter set $\epsilon_1 = \epsilon_2 = 0$, $\Delta = v$ will be called the ‘topological fully dimerized’ case, for reasons to become clear later. The energy values of the KDD in this case are $E_0 = E_1 = -v$, $E_2 = E_3 = v$. That is, the ground state is twofold degenerate. The eigenvectors and corresponding energy eigenstates have the form:

$$\psi_0 = (1, 0, 0, -1)/\sqrt{2} \equiv |e\rangle = \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle) \quad (3)$$

$$\psi_1 = (0, 1, -1, 0)/\sqrt{2} \equiv |o\rangle = \frac{1}{\sqrt{2}} (|10\rangle - |01\rangle) \quad (4)$$

$$\psi_2 = (1, 0, 0, 1)/\sqrt{2} \equiv |e'\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \quad (5)$$

$$\psi_3 = (0, 1, 1, 0)/\sqrt{2} \equiv |o'\rangle = \frac{1}{\sqrt{2}} (|10\rangle + |01\rangle) \quad (6)$$

II. FINITE ON-SITE ENERGY OPENS A MINIGAP

1. Starting from the topological fully dimerized limit, switch on identical on-site energies, $\epsilon_1 = \epsilon_2 = \epsilon$. The Hamiltonian is still diagonalized exactly; the spectrum as a function of the on-site energy is shown in Fig. 1b. The finite on-site energy ϵ splits the ground state degeneracy, i.e., creates a minigap, and the dependence of the minigap on ϵ is quadratic: $E_{\text{gap}} = o(\epsilon^2/v)$. One can consider on-site disorder: independent random but small values of the on-site energies, $\epsilon_1, \epsilon_2 \ll v$. Simple perturbation theory shows that such a disorder does open a minigap, which is also quadratic: $E_{\text{gap}} = o(\epsilon_1^2/v) + o(\epsilon_1\epsilon_2/v) + o(\epsilon_2^2/v)$. However, it is also easy to see that in the presence of weak hopping disorder or weak pair-potential disorder, the minigap is linear in the disorder.

- Exercise: Express the exact energy eigenvalues and energy eigenstates of H away from the dimerized limit, for arbitrary values of the 4 parameters.

III. THE FERMION-PARITY PSEUDOQUBIT IS PROTECTED AGAINST RELAXATION AND ON-SITE DEPHASING

- A quantum bit is a two-level quantum system. Often a selected pair of basis states are denoted $|\uparrow\rangle$ and $|\downarrow\rangle$. Pauli operators are defined using the basis states in the usual way, e.g., $\sigma_z = |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|$.
- A few features of a quantum bit, in general:
 - It can be prepared in pure superpositions, $\alpha|\uparrow\rangle + \beta|\downarrow\rangle$. If the Hamiltonian is set to zero, then the state does not evolve in time and therefore the prepared quantum state survives forever.
 - The process called ‘qubit relaxation’ takes place, e.g., in the following situation. Assume that the qubit dynamics is governed by the Hamiltonian $H(t) = B_0\sigma_z + \xi(t)\sigma_z$, where the first term is an intentional one whereas the second one is a noisy one. In this case, starting from $|\uparrow\rangle$ for example, the qubit undergoes time evolution during which the two basis states get superposed (for a certain noise realization), or mixed (if we average over noise realizations); in other words, the qubit can ‘flip’ or ‘relax’.
 - ‘Qubit dephasing’ happens, e.g., if the qubit is subject to a Hamiltonian $H(t) = \xi(t)\sigma_z$ with $\xi(t)$ being a noise, and the time evolution starts from a superposition of the two basis states, e.g., $(|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$. In that case, the Bloch vector of the qubit becomes randomized along the equator, meaning that the noise-averaged Bloch vector approaches the origin. (This is randomization is visualised in step 3 of the bottom path of Fig. 1f.) The characteristic time scale of this process is often denoted T_2^* , and is called inhomogeneous dephasing time. In the first lecture, we have seen the example of a charge qubit with hopping noise, with noise strength σ , where the inhomogeneous dephasing time was $T_{2,c}^* = \hbar/\sigma$. Similarly, a charge qubit with independent noise on the on-site energies with noise strength σ , in the absence of hopping noise, has the same inhomogeneous dephasing time $T_{2,c}^* = \hbar/\sigma$. Exercise: prove this.
- One could imagine using the even and odd ground states of the topological fully dimerized KDD as a qubit. As we argue below, one cannot prepare a superposition though; hence we will call this two-level system the ‘fermion-parity pseudoqubit’, FPPQ.
 - It is not possible to prepare a superposition of $|e\rangle$ and $|o\rangle$. Argument 1: As long as we consider the KDD itself, the even and odd sectors are decoupled. Argument 2: We could allow for an environment that helps us preparing a superposition of the two KDD ground states. In that case, the fermion parity of the full system (KDD+environment) has a well-defined fermion parity, e.g., even, at the beginning of the preparation procedure. At the end of the procedure, the fermion parity of the full system should still be even. To prepare the desired superposition, one aims at a state with the structure $\frac{1}{\sqrt{2}}(|e\rangle + |o\rangle) \otimes (\alpha|e\rangle_{\text{env}} + \beta|o\rangle_{\text{env}})$. Clearly, this is not an even state of the full system for any value of α and β , so it cannot be prepared.
 - If the KDD is isolated electronically (no tunneling) from its environment, then its fermion parity number is conserved. This means that this pseudoqubit shows no relaxation. Electrons tunneling between the KDD and its environment, or quasiparticles created, e.g., thermally, can lead to relaxation though; the related processes are often called ‘quasiparticle poisoning’.
 - The FPPQ enjoys a certain degree of protection against dephasing due to on-site disorder, if we compare the corresponding inhomogeneous dephasing time to that of a charge qubit experiencing the same amount of on-site disorder. Following the arguments above, the minigap of the FPPQ in the presence of on-site disorder of strength $\sigma \ll v$ scales as $E_{\text{gap}} \sim \sigma^2/v$. The effective Hamiltonian of the FPPQ is $H = \frac{E_{\text{gap}}}{2}\sigma_z$, which implies $T_2^* \sim \hbar/E_{\text{gap}} \sim (v/\sigma)T_{2,c}^* \gg T_{2,c}^*$, i.e., the dephasing time is indeed prolonged with respect to that of the charge qubit. Note that the FPPQ in the KDD is not protected against hopping disorder or pair-potential disorder, since the minigap scales linearly with those.

IV. THE FERMION-PARITY PSEUDOQUBIT CAN BE READ OUT AND INITIALIZED USING PARITY-TO-CHARGE CONVERSION

- The task is the following: we know that the topological fully dimerized KDD is either in $|e\rangle$ or in $|o\rangle$, and would like to find out which one. Can we do this by simply measuring a charge? It seems we cannot, because the charge

expectation values are the same in the two states, $\langle e|c_1^\dagger c_1|e\rangle = \langle e|c_2^\dagger c_2|e\rangle = \langle o|c_1^\dagger c_1|o\rangle = \langle o|c_2^\dagger c_2|o\rangle = 1/2$, which can be directly calculated from the energy eigenstates listed above. Therefore, a general strategy to distinguish between $|e\rangle$ and $|o\rangle$ is to convert the parity information to charge information first, and then measure the charge. An example arrangement that allows to realize this is depicted in Fig. 1c. Here, site 3 is an auxiliary site that is used for the charge measurement, and the orange lines denote time-dependent hopping amplitudes $u(t)$. We assume that $u(t)$ is a weak square pulse, that is, $u(t) = u_0\Theta(t)\Theta(\tau_u - t)$ and $u_0 \ll 2v$.

2. The Hamiltonian reads

$$H = H_0 + H_1 \equiv v(c_1^\dagger c_2 + c_1^\dagger c_2^\dagger + h.c.) + u(t)(c_1^\dagger c_3 + c_2^\dagger c_3 + h.c.). \quad (7)$$

The protocol goes as follows. At $t = 0$, the FPPQ formed by sites 1 and 2 are either in $|e\rangle$ or $|o\rangle$, and site 3 is empty. Then, we apply the symmetric hopping pulse for a time $\tau_u = h/(4u_0)$. After the pulse, we measure the charge on site 3. If the measured charge is zero (one), then we conclude that the parity of the initial state was odd (even).

3. The sketch of the proof is as follows. The energy spectrum and eigenstates of the decoupled system H_0 are known. The ground state is fourfold degenerate, at energy $-v$, with the four states being $|e0\rangle$, $|e1\rangle$, $|o0\rangle$, $|o1\rangle$, where the number denotes the occupation of site 3. The other four energy eigenstates are at energy v . Since the hopping pulse is weak, $u_0 \ll 2v$, we can consider the dynamics to be constrained to the ground-state subspace, characterized by the projector P . To describe that, we project the Hamiltonian to that subspace; the result expressed in the basis $|e0\rangle$, $|e1\rangle$, $|o0\rangle$, $|o1\rangle$ reads:

$$PHP = u(t) \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad (8)$$

This result reveals that $|e0\rangle$ evolves to $|o1\rangle$, but $|o0\rangle$ is a stationary state (within this low-energy approximation) even if the hopping u is switched on.

4. Note that the fact that the initial state $|o0\rangle$ does not create an electron transfer to site 3 can be interpreted as a destructive interference of the two hopping paths $1 \rightarrow 3$ and $2 \rightarrow 3$. In other words, it is a consequence of the minus sign in $|o\rangle = (|10\rangle - |01\rangle)/\sqrt{2}$ and the perfect symmetry of the two $u(t)$ pulses.
5. Note that the measurement procedure described above also serves as an initialization procedure: for both initial states $|e0\rangle$ and $|o0\rangle$, the KDD ends up in the state $|o\rangle$ after the charge measurement.
6. Dephasing does not cause errors in parity readout. Relaxation, due to quasiparticle poisoning, does cause readout errors.
7. Exercise: Assume that quasiparticle poisoning is a Poisson process: when the KDD is tuned to one topological fully dimerized case, then the ground state switches between $|e\rangle$ and $|o\rangle$ randomly, and these switches are distributed in time according to a Poisson process with time constant T_q . Using the initialization and readout tools above, devise an ‘experimental’ method to determine T_q . Assume that the quasiparticle poisoning time is long, $T_q \gg T_r$, where T_r is the time required for the initialization or readout of the FPPQ.
8. A remark (thanks to Peter Makk): For a KDD, the fermion parity can be read out without parity-to-charge conversion: in fact, by applying a strong charge measurement on both sites we obtain the total charge and thereby the fermion parity. How does that opportunity change if we take a long chain? Is it enough to measure the charge at the two end sites of the chain? No. By inspecting the ground-state wave functions for longer chains, we conclude that the fermion parity is revealed only if all local charges are measured.

V. TWO KITAEV DOUBLE DOTS FORM A SINGLE MAJORANA QUBIT

1. A qubit can be defined using two KDDs, which are uncoupled from each other, as depicted in Fig. 1d. E.g., one can choose the ‘globally even’ ground states, denoted as $|ee\rangle$ and $|gg\rangle$, as the qubit basis states. We will call this the ‘Majorana qubit’.

2. In the generic qubit example given above, a Hamiltonian $H = B(t)\sigma_z$ with the square control pulse $B(t) = B_0\Theta(t)\Theta(\tau_B - t)$ can be used to perform a rotation of the Bloch vector around the z axis with angle $\alpha = 2B_0t/\hbar$. Rotations around other axes are analogous.
3. For the Majorana qubit, an x rotation can be performed by a square control pulse $u(t)$ of the hopping between sites 2 and 3, see Fig. 1e. To see this, first we write the Hamiltonian as

$$H = H_0 + H_1(t) = v(c_1^\dagger c_2 + c_1^\dagger c_2^\dagger + c_3^\dagger c_4 + c_3^\dagger c_4^\dagger + h.c.) + u(t)(c_2^\dagger c_3 + h.c.), \quad (9)$$

and again consider the dynamics within the relevant low-energy ‘qubit subspace’ (projector: P_q) of H_0 , spanned by $|ee\rangle$ and $|oo\rangle$,

$$H_q(t) = P_q H_1(t) P_q = -\frac{1}{2}u(t)\sigma_x. \quad (10)$$

That is, a pulse of length t will cause a qubit rotation around the x axis by $\alpha = -u_0 t/\hbar$. Using a $u_0 = 10 \mu\text{eV}$ pulse strength for a $\pi/2$ rotation, the pulse duration should be $t \approx 100$ ps.

4. A z rotation can be performed on the Majorana qubit, e.g., by applying a square pulse on the hopping on the left KDD, $H_1(t) = u(t)(c_1^\dagger c_2 + h.c.)$, resulting in the qubit Hamiltonian $H_q(t) = \frac{1}{2}u(t)\sigma_z$.
5. These single-qubit operations are not protected in any sense. In contrast, operations to be introduced in the next lecture (braiding) will be protected. Nevertheless, the non-protected operations can still be useful to demonstrate the protected nature of the Majorana qubit, as shown in the next section.
6. Presumably, the first experiments showing time-resolved measurements with Majorana qubits will use non-protected single-qubit operations, similar to those discussed above.
7. Exercise: Respecting parity conservation, we could encode one qubit using two Kitaev double dots (4 sites). How many qubits can you encode in 4 Kitaev double dots?

VI. RAMSEY-TYPE EXPERIMENT REVEALS THE PROTECTED NATURE OF THE MAJORANA QUBIT

1. In section III, we argued that FPPQ is protected against inhomogeneous dephasing caused by on-site disorder, if we compare the corresponding time scale against the inhomogeneous dephasing time of a similarly disordered charge qubit. Of course, that sounds as a meaningless claim, since the FPPQ cannot be placed in a superposition, so the dephasing time cannot be measured. However, the Majorana qubit is composed of two FPPQs, and therefore it is natural to expect that it enjoys the same kind of protection against on-site disorder as a FPPQ. An experimental test, which is a simple application of the so-called ‘Ramsey experiment’ applied widely in optics and magnetic resonance, is layed out below.
2. The Ramsey experiment is used to characterize the lifetime of a qubit. The lifetime obtained with this method is often called the inhomogeneous dephasing time. The principle of the experiment is shown in Fig. 1f. The qubit is initialized in $|\downarrow\rangle$, then flipped to the y direction with a $\pi/2$ rotation around the x axis. After a waiting time τ_{wait} , another $\pi/2$ x rotation is applied, and the z component of the qubit is measured. In the absence of noise, i.e., when $H = 0$ between the pulses, the final state is $|\uparrow\rangle$, and the measurement outcome is $\langle\sigma_z\rangle(\tau_{\text{wait}}) = 1$ for any value of the waiting time τ_{wait} . In the presence of dephasing, when the noisy Hamiltonian is $H = \xi(t)\sigma_z$ between the pulses, the noise-averaged final state typically shows an evolution toward the fully mixed state, and hence $\langle\sigma_z\rangle(\tau)$ decays from 1 to 0. In the simple Gaussian quasistatic noise model, this decay is Gaussian, $\langle\sigma_z\rangle(\tau) = e^{-(\tau/T_2^*)^2}$. Hence, from this measurement, the inhomogeneous dephasing time T_2^* can be determined.
3. From the above discussions, it is clear how to implement this measurement for the Majorana qubit. Initialization and readout is provided by the parity-to-charge conversion procedure discussed in section IV. The x rotations are provided by the hopping pulse between the two KDDs forming the Majorana qubit, see Eq. (10).