Chapter 1

The Su-Schrieffer-Heeger (SSH) model

We take a hands-on approach and get to know the basic concepts of topological insulators via a concrete system: the Su-Schrieffer-Heeger (SSH) model describes spinless fermions hopping on a one-dimensional lattice with staggered hopping amplitudes. Using the SSH model, we introduce the concepts of single-particle Hamiltonian, the difference between bulk and boundary, chiral symmetry, adiabatic equivalence, topological invariants, and bulk–boundary correspondence.

Fig. 1.1

Geometry of the SSH model. Filled (empty) circles are sites on sublattice A (B), each hosting a single state. They are grouped into unit cells: the $n = 6$th cell is circled by a dotted line. Hopping amplitudes are staggered: intracell hopping $v$ (thin lines) is different from intercell hopping $w$ (thick lines). The left and right edge regions are indicated by blue and red shaded background.

1.1 The SSH Hamiltonian

The Su-Schrieffer-Heeger (SSH) model describes electrons hopping on a chain (one-dimensional lattice), with staggered hopping amplitudes, as shown in Fig. 1.1. The chain consist of $N$ unit cells, each unit cell hosting two sites, one on sublattice $A$, and one on sublattice $B$. Interactions between the electrons are neglected, and so the dynamics of each electron is described by a single-particle Hamiltonian, of the form

$$\hat{H}_v = v \sum_{m=1}^{N} |m, B \rangle \langle m, A | + v_c.$$  \hfill (1.1)
Part 1
Introduction to topological insulators

If the bulk has nontrivial topology, then the edge has disorder-resistant bound states (‘bulk-boundary correspondence’)
SSH is a tight-binding toy model for polyacetylene

Su-Schrieffer-Heeger (SSH) model of polyacetylene

Real-space tight-binding SSH Hamiltonian:
\[
\hat{H} = v \sum_{m=1}^{N} (|m,B\rangle \langle m,A| + h.c.) + w \sum_{m=1}^{N-1} (|m+1,A\rangle \langle m,B| + h.c.).
\]

For N=4:
\[
H = \begin{pmatrix}
0 & v & 0 & 0 & 0 & 0 & 0 & 0 \\
v & 0 & w & 0 & 0 & 0 & 0 & 0 \\
0 & w & 0 & v & 0 & 0 & 0 & 0 \\
0 & 0 & v & 0 & w & 0 & 0 & 0 \\
0 & 0 & 0 & w & 0 & v & 0 & 0 \\
0 & 0 & 0 & 0 & v & 0 & w & 0 \\
0 & 0 & 0 & 0 & 0 & w & 0 & v \\
0 & 0 & 0 & 0 & 0 & 0 & v & 0 
\end{pmatrix}
\]
k-space Hamiltonian maps unit circle to complex plane

Real-space tight-binding SSH Hamiltonian:

$$\hat{H} = v \sum_{m=1}^{N} (|m,B\rangle \langle m,A| + h.c.) + w \sum_{m=1}^{N-1} (|m+1,A\rangle \langle m,B| + h.c.).$$

k-space SSH Hamiltonian:

$$H(k) = \begin{pmatrix} 0 & v + we^{-ik} \\ v + we^{ik} & 0 \end{pmatrix}$$

Brillouin zone of a 1D crystal is equivalent to the unit circle:

$$f_{v,w} : \text{unit circle} \rightarrow \mathbb{C}, k \mapsto v + we^{-ik}$$
An insulating SSH Hamiltonian has a topological invariant

k-space SSH Hamiltonian:

\[ H(k) = \begin{pmatrix} 0 & v + we^{-ik} \\ v + we^{ik} & 0 \end{pmatrix} \]

band structure, valence (-) and conduction (+) bands:

\[ E(k) = \pm |f_{v,w}(k)| = \pm |v + we^{-ik}| = \pm \sqrt{v^2 + w^2 + 2vw \cos(k)} \]
SSH parameter space has two topological phases

1. The Su-Schrieffer-Heeger (SSH) model invariant: 1) it is only well defined in the thermodynamic limit, 2) it depends on the symmetries that need to be respected. An example for a topological invariant is the winding number $n$

We know that two insulating Hamiltonians are not adiabatically equivalent if their topological invariants differ. Consider as an example two Hamiltonians corresponding to two points on different sides of the phase boundary in Fig. 1.7 of the SSH model. One might think that although there is no continuous path connecting them in the phase diagram, continuously modifying the bulk Hamiltonian by the addition of extra terms can lead to a connection between them. However, their winding numbers differ, and since winding numbers cannot change under adiabatic deformation, we know that they are not adiabatically equivalent.

Number of edge states as a topological invariant

We have seen in Sect. 1.3.2, that the number of edge states at one end of the SSH model was an integer that did not change under a specific type of adiabatic deformation. We now generalize this example.

Consider energy eigenstates at the left end of a gapped chiral symmetric one-dimensional Hamiltonian in the thermodynamic limit, i.e., with length $N \to \infty$, in an energy window from $e < E < e$, with $e$ in the bulk gap. There can be nonzero energy edge states in this energy window, and zero energy edge states as well. Each nonzero energy state has to have a chiral symmetric partner, with the state and its partner occupying the same unit cells (the chiral symmetry operator is a local unitary). The number of zero energy states is finite (because of the gap in the bulk), and they can be restricted to a single sublattice each. There are $N_A$ zero energy states on sublattice $A$, and $N_B$ states on sublattice $B$.
Part 1
Introduction to topological insulators

If the bulk has nontrivial topology, then the edge has disorder-resistant bound states (‘bulk-boundary correspondence’).
Zero intracell hopping implies zero-energy states at edges

1.3 Edge states

1.3.1 Fully dimerized limits

The SSH model becomes particularly simple in the two fully dimerized cases: if the intercell hopping amplitude vanishes and the intracell hopping is set to 1, $v = 1$, $w = 0$, or vice versa, $v = 0$, $w = 1$. In both cases the SSH chain falls apart to a sequence of disconnected dimers, as shown in Fig. 1.3.

Fig. 1.3

Fully dimerized limits of the SSH model, where the chain has fallen apart to disconnected dimers. In the trivial case (top, only intracell hopping, $v = 1$, $w = 0$), every energy eigenstate is an even or an odd superposition of two sites at the same unit cell. In the topological case, (bottom, only intercell hopping, $v = 0$, $w = 1$), dimers are between neighboring unit cells, and there is 1 isolated site per edge, that must contain one zero-energy eigenstate each, as there are no onsite potentials.

The bulk in the fully dimerized limits has flat bands.

In the fully dimerized limit, one can choose a set of energy eigenstates which are restricted to one dimer each. These consist of the even (energy $E = +1$) and odd (energy $E = -1$) superpositions of the two sites forming a dimer.

In the $v = 1$, $w = 0$ case, which we call trivial, we have $v = 1$, $w = 0$:

\[
\hat{H}(|m, \text{A}_i \pm |m, \text{B}_i) = \pm (|m, \text{A}_i \pm |m, \text{B}_i)
\] (1.19)

The bulk momentum-space Hamiltonian is independent of the wavenumber $k$.

In the $v = 0$, $w = 1$ case, which we call topological, each dimer is shared between two neighboring unit cells, $v = 0$, $w = 1$:

\[
\hat{H}(|m, \text{B}_i \pm |m+1, \text{A}_i) = \pm (|m, \text{B}_i \pm |m+1, \text{A}_i)
\] (1.20)

for $m = 1, \ldots, N$. The bulk momentum-space Hamiltonian now is

\[
\hat{H}(k) = \hat{s}_x \cos k + \hat{s}_y \sin k.
\]

In both fully dimerized limits, the energy eigenvalues are independent of the wavenumber, $E(k) = 1$. In this so-called flat-band limit, the group velocity is zero, which again shows that as the chain falls apart to dimers, a particle input into the bulk will not spread along the chain.
SSH Hamiltonians have chiral symmetry

**Definition:** a $\Gamma$ local unitary operator is a *chiral symmetry* if $\Gamma H \Gamma^\dagger = -H$

SSH Hamiltonians have chiral symmetry:

For example, $N = 4$:

$$H = \begin{pmatrix}
0 & v & 0 & 0 & 0 & 0 & 0 & 0 \\
v & 0 & w & 0 & 0 & 0 & 0 & 0 \\
w & 0 & 0 & v & 0 & 0 & 0 & 0 \\
0 & v & 0 & w & 0 & 0 & 0 & 0 \\
0 & 0 & w & 0 & v & 0 & 0 & 0 \\
0 & 0 & 0 & v & 0 & w & 0 & 0 \\
0 & 0 & 0 & 0 & w & 0 & v & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & v & 0 \\
\end{pmatrix}$$

$$\Gamma = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
\end{pmatrix}$$

**Consequences of chiral symmetry**

Up-down symmetric energy spectrum: $H \psi = E \psi$ implies $H(\Gamma \psi) = -E(\Gamma \psi)$

Finite-energy eigenstates have a ‘chiral partner’ at opposite energy

A zero-energy eigenstate might be its own chiral partner

**Example:** fully dimerized topological SSH chain
Chiral symmetry implies edge states in topological SSH

Take long fully dimerized topological SSH chain \((v=0, w=1)\).
Switch on a uniform intercell hopping \(v\).
Does the zero-energy edge state survive?
It does: its energy sticks to zero due to chiral symmetry.
The energy can leave zero only if the left and right edge states hybridize.

**bulk-boundary correspondence**
Edge states are robust against chiral-symmetric disorder

Hopping disorder (respects chiral symmetry)

On-site disorder (breaks chiral symmetry)

Hopping disorder:
- Zero-energy edge states survive disorder for $\sigma < 0.5$

On-site disorder:
- Zero-energy edge states dissolved by disorder for $\sigma > 0.5$

Parameters:
- $N_{\text{Cell}} = 10$, $w = 1$, $\nu = 0$
- Disorder strength: $\sigma$ and $\Sigma$

Energy, $E$ vs Disorder strength, $\sigma$ and $\Sigma$
SSH is one creature in the zoo of topological insulators

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Table from A. W. W. Ludwig, Physica Scripta (2016)
Part 1
Introduction to topological insulators

If the bulk has nontrivial topology, then the edge has disorder-resistant bound states (‘bulk-boundary correspondence’)

Part 2
A topological quantum memory

An almost noiseless qubit is obtained by putting together many noisy qubits
The charge qubit

two sites

\[ |L\rangle \quad \quad |R\rangle \]

single electron

\[ |\psi_0\rangle = \alpha |L\rangle + \beta |R\rangle \]

\[ H = \begin{pmatrix} \epsilon & v \\ v & -\epsilon \end{pmatrix} \]

\( v \): hopping amplitude
\( \epsilon \): on-site energy difference

To preserve the state, set \( \epsilon = v = 0 \).
Hopping noise noise erases information in the charge qubit

use initial state |L>
how well is it preserved?

\[ F(t) = \sqrt{\int_{-\infty}^{\infty} dv \, p(v) \left| \langle L | \psi^{(v)}(t) \rangle \right|^2} \]
Information can survive if transferred to a less noisy qubit

\[ \psi_0 = \alpha |L\rangle + \beta |R\rangle \]

 ideally: \[ \psi(t) = \alpha |L\rangle + \beta |R\rangle \]

\[ u = 10 \sigma, \tau_u = 0.157 \hbar/\sigma \]

memory figure of merit: fidelity plateau height (~98%)
Zero-energy SSH states are protected from hopping noise

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for \( m = 1, \ldots, N \). The bulk momentum-space Hamiltonian now is \( \hat{H}(k) = \hat{s}_x \cos k + \hat{s}_y \sin k \).

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zero-energy edge states survive hopping disorder
SSH chain with hopping noise is a good quantum memory

memory figures of merit: height & duration of fidelity plateau

noisy charge qubit

noisy SSH chain serving as a noiseless memory

all hoppings $v, w$ subject to the same amount of noise
Part 2
A topological quantum memory

An almost noiseless qubit is obtained by putting together many noisy qubits
Why do you call SSH a `topological’ quantum memory? 

1. GS can be degenerate if real-space lattice is compact
2. topology of real-space lattice => degree of GS degeneracy
3. size-protected GS degeneracy
4. (Kitaev chain, SSH: symmetry-protected GS degeneracy)
Why do you call SSH a `topological’ quantum memory?

Short version of answer #1: because the Kitaev chain is called a topological quantum bit/memory, and the SSH chain has the same properties (with particle-hole -> chiral)

Answer #2: because it is a quantum memory based on a topological insulator

Answer #3: hope more people read the abstract if `topological’ is in the title

`topological quantum memory’

`size- and symmetry-protected quantum memory’

No, really. That’s very interesting.

Please go on.
Can one realize such a topological memory?

SSH model with cold atoms
topological superconductors (see Prof. Ando’s talk)

Observation of the topological soliton state in the Su–Schrieffer–Heeger model
Eric J. Meier\textsuperscript{1}, Fangzhao An\textsuperscript{1} & Bryce Gadway\textsuperscript{1}

Non-Abelian statistics and topological quantum information processing in 1D wire networks
Jason Alicea\textsuperscript{1,*}, Yuval Oreg\textsuperscript{2}, Gil Refael\textsuperscript{1}, Felix von Oppen\textsuperscript{4} and Matthew P. A. Fisher\textsuperscript{3,5}
Is the noisy SSH memory perfect?

Yes and no:
- Fidelity plateau duration can be increased arbitrarily.
- Fidelity plateau height cannot.

Reason: noise-induced uncontrolled hybridization between 1A and 2A.
The expectation value of the minigap can be expressed as

$$
\mathbb{E}(\Delta) = \frac{2}{w^{N-1}} \left[ v \operatorname{Erf} \left( \frac{v}{\sqrt{2}\sigma} \right) + \sqrt{\frac{2}{\pi}} e^{-\frac{v^2}{2\sigma^2}} \right]^N
$$

In the clear case this can be simplified as

$$
\mathbb{E}(\Delta) = \frac{2 v^N}{w^{N-1}},
$$

whereas in the full dimerized limit, i.e. $v = 0$, we can obtain

$$
\mathbb{E}(\Delta) = \frac{2}{w^{N-1}} \left( \sqrt{\frac{2}{\pi}} \sigma \right)^N.
$$

\textbf{In[2010]}:= 2 \times (\text{Sqrt}[2 / \pi] \ 0.1)^2
\textbf{Out[2010]}= 0.0127324

\textbf{In[2011]}:= 2 \times (\text{Sqrt}[2 / \pi] \ 0.1)^5
\textbf{Out[2011]}= 6.46741 \times 10^{-6}

\textbf{In[2012]}:= 2 \times (\text{Sqrt}[2 / \pi] \ 0.1)^{10}
\textbf{Out[2012]}= 2.09137 \times 10^{-11}